Physics 319
Classical Mechanics

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Lecture 6

Angular Momentum

- For a single particle

$$\vec{l} = \vec{r} \times \vec{p}$$

- It depends on the origin of the choice of coordinates (because $\vec{r}$ does). Its time derivative is

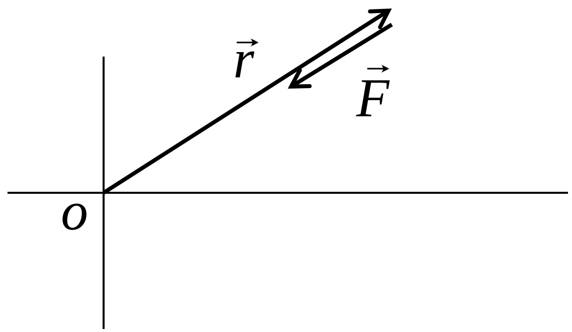
$$\frac{d\vec{l}}{dt} = \frac{d}{dt}[\vec{r} \times \vec{p}] = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}} = 0 + \vec{r} \times \vec{F}$$

- Torque

$$\vec{\Gamma} = \vec{r} \times \vec{F}$$

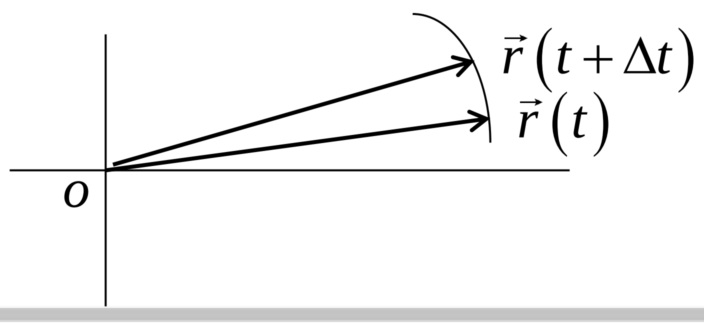
- Mostly applied when origin is the location of CM

For Central Forces to an Origin



$$\vec{r}\times\vec{F}=\vec{\Gamma}=0\rightarrow\vec{l}=\vec{r}\times\vec{p}=\vec{C}$$

- For central forces and $\vec{l}\neq 0$
  1. $\vec{r}$ and $\vec{p}$ must be in the plane perpendicular to $\vec{l}$ and the motion is in a plane
  2. Kepler's second law



$$dA=\frac{1}{2}\left|\vec{r}(t)\times\vec{v}(t)\,dt\right|$$

$$\frac{dA}{dt}=\left|\frac{\vec{l}}{2m}\right|=c=\frac{r^2(t)\dot{\theta}}{2}$$

For a system of particles

- Total angular momentum

$$\vec{L}=\sum_{\alpha}\vec{l}_{\alpha}=\sum_{\alpha}\vec{r}_{\alpha}\times\vec{p}_{\alpha}$$

$$\dot{\vec{L}}=\sum_{\alpha}\vec{r}_{\alpha}\times\dot{\vec{p}}_{\alpha}=\sum_{\alpha}\sum_{\beta\neq\alpha}\vec{r}_{\alpha}\times\vec{F}_{\alpha\beta}+\sum_{\alpha}\vec{r}_{\alpha}\times\vec{F}_{ext,\alpha}$$

$$\sum_{\alpha}\sum_{\beta\neq\alpha}\vec{r}_{\alpha}\times\vec{F}_{\alpha\beta}=\sum_{\alpha}\sum_{\beta>\alpha}\left(\vec{r}_{\alpha}\times\vec{F}_{\alpha\beta}+\vec{r}_{\beta}\times\vec{F}_{\beta\alpha}\right)=\sum_{\alpha}\sum_{\beta>\alpha}\left(\vec{r}_{\alpha}-\vec{r}_{\beta}\right)\times\vec{F}_{\alpha\beta}$$

- If force between particles central

$$\sum_{\alpha}\sum_{\beta>\alpha}\left(\vec{r}_{\alpha}-\vec{r}_{\beta}\right)\times\vec{F}_{\alpha\beta}=0$$

$$\dot{\vec{L}}=\sum_{\alpha}\vec{r}_{\alpha}\times\vec{F}_{ext,\alpha}$$

- If the net external torque on a system is zero, the system's total angular momentum is constant, *angular momentum is conserved*

Moment of Inertia

- For a rotating body, the angular momentum is proportional to the angular rotation frequency. The proportionality constant is called the *moment of inertia I*.

$$L = I\omega$$

- General expression

$$I = \sum_{\alpha} m_{\alpha} r_{\alpha}^{2}$$

- Some simple cases (from Freshman Physics)

$$I_{disc} = \frac{1}{2} mR^2 \qquad I_{rod,about\ center} = \frac{1}{12} mL^2$$

$$I_{sphere} = \frac{2}{5} mR^2 \qquad I_{rod,about\ end} = \frac{1}{3} mL^2$$

Turntable Problem